PRACTICE SET FOR MIDTERM 1

Problem 1

Solve the following integrals

$$(1) \qquad \int (x-1)\sin x \, dx.$$

We use integration by parts. Since we want the polynomial to "disappear" we choose $u = x-1$ and $dv = \sin(x)\,dx$. Therefore $du = dx$ and $v = -\cos(x)$ and

$$\int (x-1)\sin x\,dx = -\cos(x)(x-1) - \int -\cos(x)\,dx = -\cos(x)(x-1) + \int \cos(x)\,dx$$
$$= -\cos(x)(x-1) + \sin(x) + C$$

$$(2) \qquad \int \frac{\sin^3 x}{1+\cos^2 x}\,dx.$$

$$\int \frac{\sin^3(x)}{1+\cos^2(x)}\,dx = \int \frac{\sin^2(x)\sin(x)}{1+\cos^2(x)}\,dx = \int \frac{(1-\cos^2(x))\sin(x)}{1+\cos^2(x)}\,dx$$

Use the substitution $u = \cos(x)$, then $du = -\sin(x)\,dx$ and we get

$$\int \frac{(1-(\cos(x))^2)\sin(x)}{1+(\cos(x))^2}\,dx = -\int \frac{1-u^2}{1+u^2}\,du = \int \frac{u^2-1}{1+u^2}\,du = \int \frac{u^2+1-1-1}{1+u^2}\,du$$
$$= \int \frac{u^2+1}{1+u^2}\,du + \int \frac{-2}{1+u^2}\,du = \int 1\,du - 2\int \frac{1}{1+u^2}\,du$$
$$= u - 2\,\mathrm{arctg}(u) + C = \cos(x) - 2\,\mathrm{arctg}(\cos(x)) + C$$

$$(3) \qquad \int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx.$$

Use the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$ to get

$$\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx = \int \sec(u)\tan(u)\,du = \sec(u) + C = \sec(\ln x) + C.$$

$$(4) \qquad \int e^{\sqrt{x}}\,dx.$$

Use the substitution $y = \sqrt{x}$. This is the same as $y^2 = x$ which is easier to differentiate and gives $2y\,dy = dx$

$$\int e^{\sqrt{x}}\,dx = \int e^y 2y\,dy = 2\int ye^y\,dy.$$

Now we use integration by parts with $u = y$, $dv = e^y\,dy$. So $du = dy$ and $v = e^y$ and we get

$$2\int ye^y\,dy = 2\left(e^y y - \int e^y\,dy\right) = 2\left(e^y y - e^y\right) + C = 2\left(e^{\sqrt{x}}\sqrt{x} - e^{\sqrt{x}}\right) + C$$

(5) $\displaystyle\int \frac{5-2x}{x^3-4x^2+4x}\,\mathrm{d}x.$

We are going to use partial fractions. First factor the denominator:

$$x^3-4x^2+4x = x(x^2-4x+4) = x(x-2)^2.$$

Since x is a non repeated linear factor, and $x-2$ is a linear factor repeated twice, we are going to look for a decomposition of the form

$$\frac{5-2x}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Therefore

$$\begin{aligned} 5-2x+0x^2 &= A(x-2)^2 + Bx(x-2) + Cx \\ 5-2x+0x^2 &= A(x^2-4x+4) + B(x^2-2x) + Cx \\ 5-2x+0x^2 &= 4A + x(-4A-2B+C) + x^2(A+B) \end{aligned}$$

and equating the coefficients

$$\begin{cases} 4A &= 5 \\ -4A-2B+C &= -2 \\ A+B &= 0 \end{cases}$$

which means $A = 5/4$, $B = -A = -5/4$ and $C = -2+4A+2B = -2+5-5/2 = 1/2$. So we can rewrite the original integral as

$$\begin{aligned} \int \frac{5-2x}{x^3-4x^2+4x}\,\mathrm{d}x &= \frac{5}{4}\int \frac{1}{x}\,\mathrm{d}x - \frac{5}{4}\int \frac{1}{x-2}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{(x-2)^2}\,\mathrm{d}x \\ &= \frac{5}{4}\ln(|x|) - \frac{5}{4}\ln(|x-2|) + \frac{1}{2}\frac{(x-2)^{-1}}{-1} + C \end{aligned}$$

(6) $\displaystyle\int \frac{x^2+2x}{x^3-1}\,\mathrm{d}x.$

We are going to use partial fractions. First factor the denominator as a difference of cubes (see https://diegoricciotti.wordpress.com/algebra/factoring.pdf if you don't remember how to factor)

$$x^3-1 = (x-1)(x^2+x+1).$$

Since $x-1$ is a non repeated linear factor and x^2+x+1 is a non repeated irreducible quadratic factor, we look for a decomposition of the form

$$\frac{x^2+2x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

Therefore

$$\begin{aligned} x^2+2x+0 &= A(x^2+x+1) + (Bx+C)(x-1) \\ x^2+2x+0 &= Ax^2+Ax+A+Bx^2-Bx+Cx-C \\ 1x^2+2x+0 &= (A+B)x^2 + x(A-B+C) + (A-C) \end{aligned}$$

and equating the coefficients

$$\begin{cases} A+B &= 1 \\ A-B+C &= 2 \\ A-C &= 0 \end{cases}$$

which means $A = C$, $B = 1-A = 1-C$ and the second equation becomes $C-(1-C)+C = 2$,which gives $3C = 3$. In conclusion we get $C = 1$, $B = 0$ and $A = 1$, therefore the original integral becomes

$$\int \frac{x^2+2x}{x^3-1}\,\mathrm{d}x = \int \frac{1}{x-1}\,\mathrm{d}x + \int \frac{1}{x^2+x+1}\,\mathrm{d}x.$$

Now since x^2+x+1 is a quadratic irreducible, we need to complete the square and recognize it as an arctangent. Note that

$$x^2+x+1 = x^2+x+\frac{1}{4}+\frac{3}{4} = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

therefore

$$\begin{aligned}\int \frac{1}{x^2+x+1}\,\mathrm{d}x &= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x = \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right)}\,\mathrm{d}x \\ &= \frac{4}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)^2+1}\,\mathrm{d}x \\ &= \frac{4}{3}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)\frac{\sqrt{3}}{2}.\end{aligned}$$

In conclusion the original integral is equal to

$$\int \frac{x^2+2x}{x^3-1}\,\mathrm{d}x = \int \frac{1}{x-1}\,\mathrm{d}x+\int \frac{1}{x^2+x+1}\,\mathrm{d}x = \ln(|x-1|)+\frac{4}{3}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)\frac{\sqrt{3}}{2}+C.$$

(7) $$\int \frac{\sqrt[3]{x^2}-1}{x}\,\mathrm{d}x = \int \frac{\sqrt[3]{x^2}}{x}-\frac{1}{x}\,\mathrm{d}x = \int x^{2/3-1}\,\mathrm{d}x - \int \frac{1}{x}\,\mathrm{d}x = \frac{x^{2/3}}{2/3}-\ln(|x|)+C.$$

(8) $$\int (\tan^5 x+1)\sec^4 x\,\mathrm{d}x.$$

Note that $\sec^4 x = \sec^2 x \sec^2 x = (1+\tan^2(x))\sec^2 x$ and then use the substitutions $z = \tan(x)$, $\mathrm{d}z = \sec^2(x)\,\mathrm{d}x$

$$\begin{aligned}\int (\tan^5 x+1)\sec^4 x\,\mathrm{d}x &= \int ((\tan x)^5+1)(1+(\tan(x))^2)\sec^2 x\,\mathrm{d}x = \int (z^5+1)(1+z^2)\,\mathrm{d}z \\ &= \int (z^5+z^7+1+z^2)\,\mathrm{d}z = \frac{z^6}{6}+\frac{z^8}{8}+z+\frac{z^3}{3}+C \\ &= \frac{\tan^6(x)}{6}+\frac{\tan^8(x)}{8}+\tan(x)+\frac{\tan^3(x)}{3}+C\end{aligned}$$

(9) $\displaystyle\int \arcsin(2x)\,\mathrm{d}x.$

We use integration by parts with $u = \arcsin(2x)$, $\mathrm{d}v = \mathrm{d}x$ so we have $\mathrm{d}u = \frac{1}{\sqrt{1-(2x)^2}}2\,\mathrm{d}x = \frac{2}{\sqrt{1-4x^2}}$ and $v = x$

$$\int \arcsin(2x)\,\mathrm{d}x = x\arcsin(2x) - \int \frac{2x}{\sqrt{1-4x^2}}\,\mathrm{d}x.$$

Now use the substitution $y = 1-4x^2$, $\mathrm{d}y = -8x\,\mathrm{d}x$ hence $2x\,\mathrm{d}x = -\,\mathrm{d}y/4$. We focus on the integral that's left. We get

$$\int \frac{2x}{\sqrt{1-4x^2}}\,\mathrm{d}x = \int -\frac{1}{4}\frac{1}{\sqrt{y}}\,\mathrm{d}y = -\frac{1}{4}\int y^{-1/2}\,\mathrm{d}y = -\frac{1}{4}y^{1/2}2 + C = -\frac{1}{2}\sqrt{1-4x^2} + C.$$

In conclusion the original integral becomes

$$\int \arcsin(2x)\,\mathrm{d}x = x\arcsin(2x) - \int \frac{2x}{\sqrt{1-4x^2}}\,\mathrm{d}x = x\arcsin(2x) + \frac{1}{2}\sqrt{1-4x^2} + C.$$

(10) $\displaystyle\int \sin^2 x\cos^2 x\,\mathrm{d}x.$

The point is to use some trig formulas to get rid of the squares.

Method 1. Use the half-angle formulas

$$\sin^2(x) = \frac{1}{2} - \frac{\cos(2x)}{2} \qquad \cos^2(x) = \frac{1}{2} + \frac{\cos(2x)}{2}.$$

We get

$$\int \sin^2 x\cos^2 x\,\mathrm{d}x = \int \left(\frac{1}{2} - \frac{\cos(2x)}{2}\right)\left(\frac{1}{2} - \frac{\cos(2x)}{2}\right)\,\mathrm{d}x = \int \frac{1}{4} - \frac{\cos^2(2x)}{4}\,\mathrm{d}x.$$

Now we use again the half angl formula to get

$$\cos^2(2x) = \frac{1}{2} + \frac{\cos(4x)}{2}$$

hence

$$\int \frac{1}{4} - \frac{\cos^2(2x)}{4}\,\mathrm{d}x = \frac{1}{4}x - \int \left(\frac{1}{8} + \frac{\cos(4x)}{8}\right)\,\mathrm{d}x = \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\sin(4x)\frac{1}{4} + C.$$

Method 2. Use the double angle formula

$$\sin(2x) = 2\sin(x)\cos(x)$$

Divide by 2 and square both sides to get

$$\frac{\sin^2(2x)}{4} = \sin^2(x)\cos^2(x).$$

We get

$$\int \sin^2 x\cos^2 x\,\mathrm{d}x = \int \frac{\sin^2(2x)}{4}\,\mathrm{d}x = \frac{1}{4}\int \sin^2(2x)\,\mathrm{d}x = \frac{1}{4}\int \left(\frac{1}{2} - \frac{\cos(4x)}{2}\right)\,\mathrm{d}x$$
$$= \frac{1}{4}\left(\frac{1}{2}x - \frac{1}{2}\sin(4x)\frac{1}{4}\right) + C$$

(11) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,\mathrm{d}x.$

Note that $(e^{2x})' = 2e^{2x}$, so the substitution $u = e^{2x}$ would not be effective. Instead note that $e^{2x} = (e^x)^2$, hence we make the substitution $u = e^x$, so that $\mathrm{d}u = e^x\,\mathrm{d}x$

$$\int \frac{e^x}{\sqrt{1-(e^x)^2}}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \arcsin(u) + C = \arcsin(e^x) + C.$$

(12) $\displaystyle\int \frac{\ln(\arctan x)}{1+x^2}\,\mathrm{d}x.$

Use the substitution $y = \arctan(x)$, $\mathrm{d}y = \frac{1}{1+x^2}\,\mathrm{d}x$

$$\int \frac{\ln(\arctan x)}{1+x^2}\,\mathrm{d}x = \int \ln(y)\,\mathrm{d}y.$$

Now we use integration by parts with $u = \ln(y)$ and $\mathrm{d}v = \mathrm{d}y$. Then $\mathrm{d}u = \frac{1}{y}\,\mathrm{d}y$ and $v = y$

$$\begin{aligned}\int \ln(y)\,\mathrm{d}y &= y\ln(y) - \int \frac{1}{y}y\,\mathrm{d}y = y\ln(y) - \int \mathrm{d}y = y\ln(y) - y + C\\ &= \arctan(x)\ln(\arctan(x)) - \arctan(x) + C.\end{aligned}$$

(13) $\displaystyle\int e^{\frac{x}{2}}\sin x\,\mathrm{d}x.$

Here we have to integrate by parts twice. We choose $u = \sin(x)$ and $dv = e^{\frac{x}{2}}$. Then $du = \cos(x)$ and $x = 2e^{\frac{x}{2}}$. We get

$$\int e^{\frac{x}{2}}\sin(x)\,\mathrm{d}x = 2e^{\frac{x}{2}}\sin(x) - \int 2e^{\frac{x}{2}}\cos(x)\,\mathrm{d}x.$$

Now integrate by parts again, choosing $u = \cos(x)$ and $dv = 2e^{\frac{x}{2}}$. Then $du = -\sin(x)\,\mathrm{d}x$ and $v = 4e^{\frac{x}{2}}$. We get

$$\begin{aligned}\int e^{\frac{x}{2}}\sin(x)\,\mathrm{d}x &= 2e^{\frac{x}{2}}\sin(x) - \left(4e^{\frac{x}{2}}\cos(x) - \int 4e^{\frac{x}{2}}(-\sin(x))\,\mathrm{d}x\right)\\ &= 2e^{\frac{x}{2}}\sin(x) - 4e^{\frac{x}{2}}\cos(x) - 4\int e^{\frac{x}{2}}\sin(x)\,\mathrm{d}x.\end{aligned}$$

Denoting the red integral by I, the previous relation can be written as the equation

$$\begin{aligned} I &= 2e^{\frac{x}{2}}\sin(x) - 4e^{\frac{x}{2}}\cos(x) - 4I\\ I + 4I &= 2e^{\frac{x}{2}}\sin(x) - 4e^{\frac{x}{2}}\\ 5I &= 2e^{\frac{x}{2}}\sin(x) - 4e^{\frac{x}{2}}\cos(x)\\ I &= \frac{1}{5}\left(2e^{\frac{x}{2}}\sin(x) - 4e^{\frac{x}{2}}\cos(x)\right).\end{aligned}$$

Therefore

$$\int e^{\frac{x}{2}}\sin(x)\,\mathrm{d}x\frac{1}{5}\left(2e^{\frac{x}{2}}\sin(x) - 4e^{\frac{x}{2}}\cos(x)\right) + C.$$

(14) $\displaystyle\int \sqrt{1-4x^2}\,\mathrm{d}x.$

We use a trigonometric substitution. A trigonometric identity involving the difference of squares is for example

$$1-\sin^2(t)=\cos^2(t).$$

Therefore, after noting $4x^2=(2x)^2$ we need to set $2x=\sin(t)$. We get $2\,\mathrm{d}x=\cos(t)\,\mathrm{d}t$ hence $\mathrm{d}x=\frac{1}{2}\cos(t)\,\mathrm{d}t$

$$\begin{aligned}\int \sqrt{1-(2x)^2}\,\mathrm{d}x &= \frac{1}{2}\int \sqrt{1-\sin^2(t)}\cos(t)\,\mathrm{d}t = \frac{1}{2}\int \cos^2(t)\,\mathrm{d}t = \frac{1}{2}\int \left(\frac{1}{2}+\frac{\cos(2t)}{2}\right)\mathrm{d}t \\ &= \frac{1}{2}\left(\frac{1}{2}t+\frac{1}{2}\frac{\sin(2t)}{2}\right)+C.\end{aligned}$$

Now to convert to the original variable x we look at the substituion we made $2x=\sin(t)$. Therefore $t=\arcsin(2x)$. In order to substitute $\sin(2t)$ we use the double angle formula $\sin(2t)=2\sin(t)\cos(t)$. We have $\cos(t)=\sqrt{1-\sin^2(t)}=\sqrt{1-4x^2}$ so we conclude

$$\frac{1}{2}\left(\frac{1}{2}t+\frac{1}{2}\frac{\sin(2t)}{2}\right)+C=\frac{1}{2}\left(\frac{1}{2}\arcsin(2x)+\frac{1}{2}\arcsin(2x)\sqrt{1-4x^2}\right)+C.$$

Alternatively you can set up a right triangle and do it geometrically.

(15) $\displaystyle\int x^2 e^{-3x^3}\,\mathrm{d}x.$

We use the substitution $u=-3x^3$, $\mathrm{d}u=-9x^2\,\mathrm{d}x$ hence $x^2\,\mathrm{d}x=-\frac{1}{9}\,\mathrm{d}u$

$$\int x^2 e^{-3x^3}\,\mathrm{d}x=-\frac{1}{9}\int e^u\,\mathrm{d}u=-\frac{1}{9}e^u+C=-\frac{1}{9}e^{-3x^3}+C$$

(16) $\displaystyle\int (x^3-1)\ln x\,\mathrm{d}x.$

Here we use integration by parts. Since we don't know an immediate antiderivative of $\ln(x)$ we choose $u=\ln(x)$ and $dv=(x^3-1)\,\mathrm{d}x$. This way $\mathrm{d}u=\frac{1}{x}\,\mathrm{d}x$ and $v=\frac{x^4}{4}-x$

$$\begin{aligned}\int (x^3-1)\ln x\,\mathrm{d}x &= \left(\frac{x^4}{4}-x\right)\ln(x)-\int\left(\frac{x^4}{4}-x\right)\frac{1}{x}\,\mathrm{d}x \\ &= \left(\frac{x^4}{4}-x\right)\ln(x)-\int\left(\frac{x^3}{4}-1\right)\mathrm{d}x \\ &= \left(\frac{x^4}{4}-x\right)\ln(x)-\left(\frac{x^4}{16}-x\right)\mathrm{d}x+C\end{aligned}$$

(17) $\displaystyle\int x \arctan(1+x)\,\mathrm{d}x.$

We could directly use an integration by parts, but it will be simpler to simplify the argument of the arctangent first. So we use the substituion $y = 1 + x$, $\mathrm{d}y = \mathrm{d}x$.

$$\int x \arctan(1+x)\,\mathrm{d}x = \int (y-1)\,\mathrm{arctg}(y)\,\mathrm{d}y.$$

Now we use integration by parts. Since we don't know an immediate antiderivative of $\mathrm{arctg}(y)$ we choose $u = \mathrm{arctg}(y)$ and $\mathrm{d}v = (y-1)\,\mathrm{d}y$. We get $\mathrm{d}u = \frac{1}{1+y^2}\,\mathrm{d}y$ and $v = \frac{y^2}{2} - y$

$$\int (y-1)\,\mathrm{arctg}(y)\,\mathrm{d}y = \left(\frac{y^2}{2} - y\right)\mathrm{arctg}(y) - \int \left(\frac{y^2}{2} - y\right)\frac{1}{1+y^2}\,\mathrm{d}y.$$

We now focus on the last integral

$$\begin{aligned}\int \left(\frac{y^2}{2} - y\right)\frac{1}{1+y^2}\,\mathrm{d}y &= \frac{1}{2}\int \frac{y^2}{1+y^2}\,\mathrm{d}y - \int \frac{y}{1+y^2}\,\mathrm{d}y \\ &= \frac{1}{2}\int \frac{y^2+1-1}{1+y^2}\,\mathrm{d}y - \int \frac{y}{1+y^2}\,\mathrm{d}y \\ &= \frac{1}{2}\left(\int \frac{y^2+1}{1+y^2}\,\mathrm{d}y - \int \frac{1}{1+y^2}\,\mathrm{d}y\right) - \int \frac{y}{1+y^2}\,\mathrm{d}y \\ &= \frac{1}{2}\left(y - \mathrm{arctg}(y)\right) - \ln(1+y^2)\frac{1}{2}\end{aligned}$$

where we solved the last integral noting that the numerator is the derivative of the denominator except for a factor of 2 (you can check it using the substitution $t = 1 + y^2$). Now going back to the original variable we get

$$\begin{aligned}\int x \arctan(1+x)\,\mathrm{d}x &= \left(\frac{y^2}{2} - y\right)\mathrm{arctg}(y) - \int \left(\frac{y^2}{2} - y\right)\frac{1}{1+y^2}\,\mathrm{d}y \\ &= \left(\frac{y^2}{2} - y\right)\mathrm{arctg}(y)\frac{1}{2}\left(y - \mathrm{arctg}(y)\right) + \ln(1+y^2)\frac{1}{2} + C \\ &= \left(\frac{(1+x)^2}{2} - (1+x)\right)\mathrm{arctg}(1+x)\frac{1}{2}\left(1 + x - \mathrm{arctg}(1+x)\right) \\ &\quad + \ln(1+(1+x)^2)\frac{1}{2} + C.\end{aligned}$$

(18) $\displaystyle\int_1^\infty \frac{1}{2x^2+x-1}\,\mathrm{d}x.$

This is clearly an improper integral since one of the bounds of integration is infinity. The function $2x^2+x-1$ is equal to zero when $x = \frac{-1\pm\sqrt{1+8}}{4} = \frac{-1\pm 3}{4}$, so for $x=-1$ or $x=1/2$. Our interval of integration doesn't contain either of them, so our function is continuous on $[1,\infty[$ and the only problem is at ∞. By definition we have

$$\int_1^\infty \frac{1}{2x^2+x-1}\,\mathrm{d}x = \lim_{M\to\infty}\int_1^M \frac{1}{2x^2+x-1}\,\mathrm{d}x.$$

First we are going to compute the indefinite integral. We use a partial fractions decomposition. Since the denomiator factors as $2x^2+x-1 = 2(x+1)(x-1/2) = (x+1)(2x-1)$ which are linear factors appearing only once, we look for a decomposition of the form

$$\frac{1}{2x^2+x-1} = \frac{1}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}$$

hence

$$1 = A(2x-1) + B(x+1).$$

Choose $x=-1$ to get $1 = A(-3)$ which means $A = -\frac{1}{3}$. Then choose $x = 1/2$ to get $1 = B3/2$ which means $B = 2/3$. Therefore we have

$$\begin{aligned}\int \frac{1}{2x^2+x-1}\,\mathrm{d}x &= -\frac{1}{3}\int\frac{1}{x+1}\,\mathrm{d}x + \frac{2}{3}\int\frac{1}{2x-1}\,\mathrm{d}x\\ &= -\frac{1}{3}\ln|x+1| + \frac{2}{3}\ln|2x-1|\frac{1}{2}\\ &= \frac{1}{3}\left(-\ln(|x+1|) + \ln(|2x-1|)\right)\\ &= \frac{1}{3}\ln\left(\frac{|2x-1|}{|x+1|}\right)\end{aligned}$$

where we used properties of logarithms in the last step to simplify the result. We are doing this because we are going to compute the limit now.

$$\begin{aligned}\int_1^\infty \frac{1}{2x^2+x-1}\,\mathrm{d}x &= \lim_{M\to\infty}\int_1^M \frac{1}{2x^2+x-1}\,\mathrm{d}x\\ &= \lim_{M\to\infty}\frac{1}{3}\ln\left(\frac{|2M-1|}{|M+1|}\right) - \frac{1}{3}\ln(1)\\ &= \lim_{M\to\infty}\frac{1}{3}\ln\left(\frac{2M-1}{M+1}\right)\end{aligned}$$

since $\frac{2M-1}{M+1}$ is positive for M large, and $\ln(1)=0$. Now note that

$$\lim_{M\to\infty}\frac{2M-1}{M+1} = 2$$

hence

$$\lim_{M\to\infty}\frac{1}{3}\ln\left(\frac{2M-1}{M+1}\right) = \frac{1}{3}\ln(2)$$

whichis a finite value, therefore the integral is convergent.

(19) $\displaystyle\int_0^\infty \frac{1}{5+x^2}\,\mathrm{d}x.$

The denominator $5+x^2$ is never equal to zero, so the integrand is continuous on the interval $[0,\infty[$. The integral is improper only at infinity. By definition we have

$$\begin{aligned}\int_0^\infty \frac{1}{5+x^2}\,\mathrm{d}x &= \lim_{M\to\infty}\int_0^M \frac{1}{5+x^2}\,\mathrm{d}x = \lim_{M\to\infty}\frac{1}{5}\int_0^M \frac{1}{1+\frac{x^2}{5}}\,\mathrm{d}x \\ &= \lim_{M\to\infty}\frac{1}{5}\int_0^M \frac{1}{1+\left(\frac{x}{\sqrt{5}}\right)^2}\,\mathrm{d}x \\ &= \lim_{M\to\infty}\frac{1}{5}\left[\operatorname{arctg}\left(\frac{x}{\sqrt{5}}\sqrt{5}\right)\right]_0^M \\ &= \lim_{M\to\infty}\frac{1}{5}\operatorname{arctg}\left(\frac{M}{\sqrt{5}}\sqrt{5}\right)\frac{1}{5}\frac{\pi}{2}\end{aligned}$$

therefore the integral is convergent.

(20) $\displaystyle\int_0^\infty e^{-x}\sqrt{e^{-x}+3}\,\mathrm{d}x.$

The integral is improper only at infinity. By definition we have

$$\int_0^\infty e^{-x}\sqrt{e^{-x}+3}\,\mathrm{d}x = \lim_{M\to\infty}\int_0^M e^{-x}\sqrt{e^{-x}+3}\,\mathrm{d}x.$$

Let's look at the indefinite integral first. We can make the substitution $u = e^{-x}+3$ so that $\mathrm{d}u = -e^{-x}\,\mathrm{d}x$ and we get

$$\int e^{-x}\sqrt{e^{-x}+3}\,\mathrm{d}x = -\int \sqrt{u}\,\mathrm{d}u = -u^{3/2}\frac{2}{3} + C = -(e^{-x}+3)^{3/2}\frac{2}{3} + C.$$

Therefore, going back to the improper integral we get

$$\begin{aligned}\lim_{M\to\infty}\int_0^M e^{-x}\sqrt{e^{-x}+3}\,\mathrm{d}x &= \lim_{M\to\infty}\int_0^M -(e^{-M}+3)^{3/2}\frac{2}{3} + (1+3)^{3/2}\frac{2}{3} \\ &= -(0+3)^{3/2}\frac{2}{3} + (1+3)^{3/2}\frac{2}{3}\end{aligned}$$

since the exponential approaches 0 as the exponent approaches NEGATIVE INFINITY. Therefore the integral is convergent.

(21) $\int_0^{\frac{1}{2}} \frac{1}{x\ln^2 x}\,\mathrm{d}x.$

The integrand is not defined at $x = 0$, therefore the integral is improper at 0. By definition we have

$$\int_0^{\frac{1}{2}} \frac{1}{x\ln^2 x}\,\mathrm{d}x = \lim_{t\to 0^+} \int_t^{\frac{1}{2}} \frac{1}{x\ln^2 x}\,\mathrm{d}x.$$

Let's look at the indefinite integral first. We can use the substitution $u = \ln(x)$ so that $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and we get

$$\int \frac{1}{x\ln^2 x}\,\mathrm{d}x = \int \frac{1}{u^2}\,\mathrm{d}u = -u^{-1} + C = -\frac{1}{\ln(x)} + C.$$

Going back to the improper integral we have

$$\lim_{t\to 0^+} \int_t^{\frac{1}{2}} \frac{1}{x\ln^2 x}\,\mathrm{d}x = \lim_{t\to 0^+} -\frac{1}{\ln(1/2)} + \frac{1}{\ln(t)} = -\frac{1}{\ln(1/2)}$$

because $\lim_{t\to 0^+} \ln(t) = -\infty$ so $\lim_{t\to 0^+} \frac{1}{\ln(t)} = 0$. Therefore the integral is convergent.